

# FROBENIUS BETTI NUMBERS AND MODULES OF FINITE PROJECTIVE DIMENSION

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**ABSTRACT.** Let  $(R, \mathfrak{m}, K)$  be a local ring, and let  $M$  be an  $R$ -module of finite length. We study asymptotic invariants,  $\beta_i^F(M, R)$ , defined by twisting with Frobenius the free resolution of  $M$ . This family of invariants includes the Hilbert-Kunz multiplicity ( $e_{HK}(\mathfrak{m}, R) = \beta_0^F(K, R)$ ). We discuss several properties of these numbers that resemble the behavior of the Hilbert-Kunz multiplicity. Furthermore, we study when the vanishing of  $\beta_i^F(M, R)$  implies that  $M$  has finite projective dimension. In particular, we give a complete characterization of the vanishing of  $\beta_i^F(M, R)$  for one-dimensional rings. As a consequence of our methods, we give conditions for the non-existence of syzygies of finite length.

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## 1. INTRODUCTION

Let  $(R, \mathfrak{m}, K)$  denote an  $F$ -finite local ring of dimension  $d$  and characteristic  $p > 0$ , and let  $\alpha = \log_p[K : K^p]$ . Given an  $R$ -module  $M$  and an integer  $e \geq 0$ ,  ${}^eM$  denotes the  $R$ -module structure on  $M$  given by  $r * m = r^{p^e}m$  for every  $m \in {}^eM$  and  $r \in R$ . In addition,  $\lambda_R(M)$ , or simply  $\lambda(M)$  when the ring is clear from the context, denotes the length of  $M$  as an  $R$ -module.

Let  $q = p^e$  be a power of  $p$ . For an ideal  $I \subseteq R$ , let  $I^{[q]} = (i^q \mid i \in I)$  be the ideal generated by the  $q$ -th powers of elements in  $I$ . If  $I$  is  $\mathfrak{m}$ -primary, the *Hilbert-Kunz multiplicity* of  $I$  in  $R$  is defined by

$$e_{HK}(I, R) = \lim_{e \rightarrow \infty} \frac{\lambda(R/I^{[q]})}{q^d}.$$

The existence of the previous limit was proven by Monsky [Mon83]. Under mild conditions,  $e_{HK}(\mathfrak{m}, R) = 1$  if and only if  $R$  is a regular ring [WY00]. The Hilbert-Kunz multiplicity can be interpreted as a measure of singularity: the smaller it is, the nicer the ring is. For instance, Aberbach and Enescu proved rings with small Hilbert-Kunz multiplicity are Gorenstein and  $F$ -regular [AE08] (see also [BE04]). We have that

$$\lambda(R/I^{[q]}) = q^\alpha \lambda(R/I \otimes_R {}^eR) = q^\alpha \lambda(\mathrm{Tor}_0^R(R/I, {}^eR))$$

This gives rise to the following extension of the Hilbert-Kunz multiplicity for higher Tor functors. Let  $N$  be a finitely generated  $R$ -module. For an integer  $i \geq 0$  define

$$\beta_i^F(M, N) = \lim_{e \rightarrow \infty} \frac{\lambda(\mathrm{Tor}_i^R(M, {}^e N))}{q^{(d+\alpha)}}.$$

We denote  $\beta_i^F(K, R)$  by  $\beta_i^F(R)$  and call it the  $i$ -th *Frobenius Betti number* of  $R$ .

These higher invariants also detect regularity. Namely, Aberbach and Li [AL08] show that  $R$  is a regular ring if and only if  $\beta_i^F(R) = 0$  for some  $i \geq 1$ . Note that  $R$  is regular if and only if  $K$  has finite projective dimension as  $R$ -module. In this manuscript, we seek an answer to the following question.

**Question 1.1.** Let  $M$  be an  $R$ -module of finite length. What vanishing conditions on  $\beta_i^F(M, R)$  imply that  $M$  has finite projective dimension?

Miller [Mil00] showed if  $R$  is a complete intersection and  $M$  is an  $R$ -module of finite length, then the vanishing of  $\beta_i^F(M, R)$  for some  $i \geq 1$  implies that  $M$  has finite projective dimension. We refer to [DS13] for related results for Gorenstein rings. In Section 4, we answer this question for rings that have small regular algebras, and for rings that have  $F$ -contributors. Later, we focus on one-dimensional rings and give the following characterization for the vanishing of  $\beta_i^F(M, R)$ .

**Theorem.** (see Theorem 4.7) Let  $(R, \mathfrak{m}, K)$  be a one-dimensional local ring of positive characteristic  $p$ , and let  $M$  be an  $R$ -module of finite length. Let  $(G_j, \varphi_j)_{j \geq 0}$  be a minimal free resolution of  $M$ . Then the following are equivalent:

- (i)  $\mathrm{Im}(\varphi_{i+1}) \subseteq H_{\mathfrak{m}}^0(G_i)$ .
- (ii)  $\mathrm{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$  for all  $e \geq 0$ , for all  $\mathfrak{p} \in \mathrm{Min}(R)$ .
- (iii)  $\mathrm{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$  for all  $e \gg 0$ , for all  $\mathfrak{p} \in \mathrm{Min}(R)$ .
- (iv)  $\beta_i^F(M, R) = 0$ .

Assume in addition that  $R$  is complete and  $K$  is algebraically closed. If  $V$  denotes the integral closure of  $R$  in its ring of fractions, then the conditions above are equivalent to

- (v)  $\mathrm{Tor}_i^R(M, V) = 0$ .

As a consequence of this theorem, we show that if  $R$  is a one-dimensional Cohen-Macaulay local ring and  $\lambda(M) < \infty$ , then  $\beta_i^F(M, R) = 0$  for any  $i \geq 1$  implies that  $M$  has finite projective dimension (see Corollary 4.8). Furthermore, we prove that the vanishing of two consecutive  $\beta_i^F(M, R)$  implies that  $M$  has finite projective dimension in every one-dimensional local ring (see Corollary 4.9).

From the theorem above we have that  $\beta_i^F(M, R) = 0$  if and only if the  $(i+1)$ -syzygy has finite length. On the other hand, there are modules of infinite projective dimension over one-dimensional rings which have second syzygies of finite length (see Example 5.1). Motivated by Iyengar's question about the eventual stability of dimensions of syzygies and by our results regarding  $\beta_i^F(M, R)$ , we ask the following question.

**Question 1.2.** Let  $R$  be a  $d$ -dimensional local ring, and let  $M$  be a finitely generated  $R$ -module such that  $\mathrm{pd}_R(M) = \infty$  and  $\lambda(M) < \infty$ . If  $i > d + 1$ , then must the length of the  $i$ -th syzygy be infinite?

In Section 5, we study this question, mainly for one-dimensional rings. In particular, we show that the answer to Question 1.2 is positive for one-dimensional Buchsbaum rings (see

Proposition 5.3). We also obtain a partial answer for modules whose Betti numbers are eventually non-decreasing (see Proposition 5.7). Furthermore, we show that the first and third syzygies of  $M$  are either zero or have infinite length for every finite length module  $M$  over a one-dimensional ring (see Corollary 5.10). The assumption of  $M$  having finite length is necessary, as shown in Example 5.11. Aside from the study of projective dimension, we study basic properties of the higher invariants that resemble the Hilbert-Kunz multiplicity in other aspects.

## 2. NOTATION AND TERMINOLOGY

Throughout this article,  $(R, \mathfrak{m}, K)$  will denote a local ring of Krull dimension  $\dim(R) = d$ . For a finitely generated  $R$ -module  $M$ , we define  $\dim(M) = \dim(R/(0 :_R M))$ , where  $0 :_R M = \{x \in R \mid xM = 0\}$ . When  $M = 0$ , we set  $\dim(M) = -1$ . An element  $x \in R$  such that  $\dim(R/(x)) = d - 1$  will be called a *parameter* of  $R$ . Given a finitely generated  $R$ -module  $M$ , a *minimal free resolution*  $(G_\bullet, \varphi_\bullet)$  of  $M$  is an exact sequence

$$\dots \longrightarrow G_{i+1} \xrightarrow{\varphi_{i+1}} G_i \xrightarrow{\varphi_i} \dots \longrightarrow G_1 \xrightarrow{\varphi_1} G_0 \longrightarrow M \longrightarrow 0$$

such that  $G_i \cong R^{\beta_i(M)}$  are free  $R$ -modules and  $\text{Im}(\varphi_{i+1}) \subseteq \mathfrak{m}G_i$ . The integers  $\beta_i(M) = \text{rk}(G_i) = \lambda(\text{Tor}_j^R(M, K))$  are called the *Betti numbers* of  $M$ . If  $\beta_i(M) = 0$  for some  $i$ , we say that  $M$  has *finite projective dimension*, and that it is equal to  $\text{pd}_R(M) = \max\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$ . We adopt the convention that  $\text{pd}_R(M) = -\infty$ , when  $M = 0$ . For all  $i \geq 0$  we set  $\Omega_i(M) = \text{Coker}(\varphi_i)$ , and we call it the  *$i$ -th syzygy of the module  $M$* . Note that  $\Omega_0(M) = M$ . When no confusion may arise, we will denote  $\Omega_i(M)$  simply by  $\Omega_i$ .

Throughout the manuscript we often make use of local cohomology tools. For every  $k \in \mathbb{N}$ , the quotient map  $R/\mathfrak{m}^{k+1} \rightarrow R/\mathfrak{m}^k$  induces maps of functors

$$\text{Ext}_R^i(R/\mathfrak{m}^k, -) \rightarrow \text{Ext}_R^i(R/\mathfrak{m}^{k+1}, -).$$

For an  $R$ -module  $M$ , we define the  *$i$ -th local cohomology of  $M$  with support on  $\mathfrak{m}$*  by

$$H_{\mathfrak{m}}^i(M) = \varinjlim_{k \rightarrow \infty} \text{Ext}_R^i(R/\mathfrak{m}^k, M).$$

In particular,  $H_{\mathfrak{m}}^0(M) = \bigcup_{k \in \mathbb{N}} 0 :_M \mathfrak{m}^k = \{v \in M \mid \mathfrak{m}^k v = 0 \text{ for some } k \in \mathbb{N}\}$ . For a non-zero finitely generated  $R$ -module  $M$ ,  $\text{depth}(M)$  denotes the smallest integer  $j$  such that  $H_{\mathfrak{m}}^j(R) \neq 0$ . When  $\text{depth}(M) = \dim(M)$ , the module is called *Cohen-Macaulay*, and  $M$  is called *maximal Cohen-Macaulay* if  $\text{depth}(M) = \dim(R)$ .

We now review some basic facts about integral closures. For an ideal  $I \subseteq R$  and an element  $x \in R$  we say that  $x$  is integral over  $I$  if it satisfies an equation of the form  $x^n + r_1 x^{n-1} + \dots + r_n = 0$ , where  $r_j \in I^j$  for all  $j = 1, \dots, n$ . The set of elements integral over  $I$  forms an ideal, which is called the *integral closure* of  $I$ , and denoted  $\overline{I}$ . For an ideal  $J \subseteq I$ , we say that  $J$  is a *reduction* of  $I$  if  $\overline{J} = \overline{I}$ . We say that  $J$  is a *minimal reduction* of  $I$  if it is a reduction of  $I$  which is minimal with respect to containment. We refer the reader to [SH06, Chapter 8] for more details about reductions. For a domain  $R$ , let  $V$  be the integral closure of  $R$  in its field of fractions  $L$ . We define the *conductor* of  $R$  as the set of all elements  $z \in L$  such that  $zV \subseteq R$ , and we denote it by  $C$ . When  $V$  is finite over  $R$ , one can show that  $C$  is the largest ideal which is common to  $R$  and  $V$ , and that  $C$  contains a non-zero divisor for  $R$  [SH06, Exercise 2.11]. In particular, if  $(R, \mathfrak{m}, K)$  is an excellent one dimensional local domain, the conductor is  $\mathfrak{m}$ -primary. See [SH06, Chapter 12] for more results about conductors.

We also need the notion of dualizing complex. We refer to [Rob80, P. 51] or to [Har66, Chapter V] for more details.

**Definition 2.1.** Let  $(S, \mathfrak{n}, L)$  be a local ring of dimension  $d$ . We say that a complex  $D^\bullet$  is a dualizing complex of  $S$  if

- (1)  $D^i = \bigoplus_{\dim S/\mathfrak{p}=d-i} E_S(S/\mathfrak{p})$ .
- (2) The cohomology  $H^i(D^\bullet)$  is finitely generated.

**Remark 2.2.** If  $(S, \mathfrak{n}, L)$  is a complete ring, then  $S$  has a dualizing complex,  $D_S^\bullet$  [Har66, P. 299]. If  $\mathfrak{p}$  is a prime ideal such that  $\dim S/\mathfrak{p} = \dim S$ , we have that  $S_{\mathfrak{p}}$  is Artinian, hence complete. In addition,  $D_{S_{\mathfrak{p}}}^\bullet := D_S^\bullet \otimes S_{\mathfrak{p}}$  is a dualizing complex for  $S_{\mathfrak{p}}$ . Furthermore,  $H^j(D_{S_{\mathfrak{p}}}^\bullet) = H^j(D_S^\bullet) \otimes S_{\mathfrak{p}} = 0$  for  $j > 0$  and  $\omega_{S_{\mathfrak{p}}} \cong H^0(D_{S_{\mathfrak{p}}}^\bullet) = E_{S_{\mathfrak{p}}}(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}})$ , because  $S_{\mathfrak{p}}$  is Artinian, and so, Cohen-Macaulay.

We now introduce Buchsbaum rings, which we study Question 1.2 in Section 5.

**Definition 2.3.** Let  $(R, \mathfrak{m}, K)$  be a local ring of dimension  $d$ . We say that  $R$  is a *Buchsbaum ring* if, for any system of parameters  $x_1, \dots, x_d$ , we have

$$(x_1, \dots, x_{i-1}) : x_i = (x_1, \dots, x_{i-1}) : \mathfrak{m}$$

for every  $i = 1, \dots, d$ . When  $i = 1$ , the ideal  $(x_1, \dots, x_{i-1})$  is simply the zero ideal.

There are several equivalent ways to define Buchsbaum rings, but the one above is the most convenient for our purposes.

**Remark 2.4.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional local ring. Suppose that  $R$  is not Cohen-Macaulay, so that  $H_{\mathfrak{m}}^0(R) \neq 0$ . Then there exists a parameter  $x$  of  $R$  such that  $H_{\mathfrak{m}}^0(R) = 0 :_R x$ . In fact, fix an integer  $n \in \mathbb{N}$  such that  $\mathfrak{m}^n H_{\mathfrak{m}}^0(R)$ , using that  $H_{\mathfrak{m}}^0(R) \subseteq R$  is an ideal, hence it is finitely generated. Take any parameter  $y \in \mathfrak{m}$ , and set  $x = y^n$ . With this choice, one has  $x H_{\mathfrak{m}}^0(R) \subseteq \mathfrak{m}^n H_{\mathfrak{m}}^0(R) = 0$ , so that  $H_{\mathfrak{m}}^0(R) \subseteq 0 :_R x$ . On the other hand, there exists  $k \in \mathbb{N}$  such that  $\mathfrak{m}^k \subseteq (x)$ . Therefore, if  $r \in 0 :_R x$ , we get  $r \mathfrak{m}^k \subseteq r(x) = 0$ , so that  $r \in H_{\mathfrak{m}}^0(R)$ . We conclude that  $H_{\mathfrak{m}}^0(R) = 0 :_R x$ .

**Remark 2.5.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional Buchsbaum ring. By Remark 2.4, there exists a parameter  $x \in R$  such that  $0 :_R x = H_{\mathfrak{m}}^0(R)$ . By definition of Buchsbaum ring, we have that

$$H_{\mathfrak{m}}^0(R) = 0 :_R x = 0 :_R \mathfrak{m}.$$

In particular,  $\mathfrak{m} H_{\mathfrak{m}}^0(R) = 0$ , that is,  $H_{\mathfrak{m}}^0(R) \cong \bigoplus_{j=1}^t K$  is a finite dimensional  $K$ -vector space.

For the rest of the section, we assume that  $(R, \mathfrak{m}, K)$  is a local ring of characteristic  $p > 0$ . For an integer  $e \geq 1$ , we consider the  $e$ -th iteration of the Frobenius endomorphism  $F^e : R \rightarrow R$ ,  $F^e(r) = r^{p^e}$  for all  $r \in R$ . For an  $R$ -module  $M$ , one can consider  $M$  with the action induced by restriction of scalars, via  $F^e$ . We denote this module by  ${}^e M$ . More explicitly, for  $r \in R$  and  $m \in {}^e M$ , we have  $r * m = r^{p^e} m$ .

**Definition 2.6.** We say that  $R$  is *F-finite* if  ${}^1 R$  is a finitely generated  $R$ -module.

Note that  $R$  is *F-finite* if and only if  ${}^e R$  is a finitely generated  $R$ -module for any  $e \geq 1$  or, equivalently, for all  $e \geq 1$ . Furthermore, *F-finite* rings are excellent [Kun76, Theorem 2.5]. When  $R$  is *F-finite*, we have that  $[K : K^p] < \infty$ . In this case, we set  $\alpha = \log_p [K : K^p]$ .

3. DEFINITION AND PROPERTIES OF  $\beta_i^F(M, N)$  AND  $\mu_i^F(M, N)$ 

We start by defining the Frobenius Betti numbers and showing basic properties that resemble the Hilbert-Kunz multiplicity.

**Definition 3.1** (see also [Li08]). Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ , let  $M$  be an  $R$ -module of finite length and let  $N$  be a finitely generated  $R$ -module. Define

$$\beta_{i,R}^F(M, N) = \lim_{e \rightarrow \infty} \frac{\lambda(\mathrm{Tor}_i^R(M, {}^eN))}{q^{(d+\alpha)}}.$$

We denote  $\beta_{i,R}^F(K, R)$  by  $\beta_{i,R}^F(R)$  and call it the  $i$ -th Frobenius Betti number of  $R$ . If the ring is clear from the context, we only write  $\beta_i^F(M, N)$ . The above limit exists by the main result in [Sei89].

We point out that Li [Li08] focused on  $\beta_i^F(R/I, R)$ , which he denoted by  $t_i(I, R)$ .

**Example 3.2.** Suppose that  $R = S/fS$ , where  $S$  is an  $F$ -finite regular local ring of characteristic  $p > 0$ , and  $f \in S$ . We can write  ${}^eR \cong R^{a_e} \oplus M_e$ , where  $M_e$  has no free summands. The limit  $s(R) := \lim_{e \rightarrow \infty} \frac{a_e}{q^{(d+\alpha)}}$  exists [Tuc12, Theorem 4.9], and it is called the  $F$ -signature of  $R$ , which is an important invariant related to strong  $F$ -regularity [AL03, Theorem 0.2]. We consider the minimal free resolution of  ${}^eR$ :

$$\dots \longrightarrow R^{\beta_i({}^eR)} \longrightarrow R^{\beta_{i-1}({}^eR)} \longrightarrow \dots \longrightarrow R^{\beta_0({}^eR)} \longrightarrow {}^eR \longrightarrow 0.$$

We note that  $\beta_0({}^eR) = a_e + \beta_0(M_e)$  and  $\beta_i({}^eR) = \beta_i(M_e)$  for  $i > 0$ . Since  $M_e$  is a maximal Cohen-Macaulay module with no free summands, we have that  $\beta_i(M_e) = \beta_0(M_e)$  for  $i > 0$  [Eis80, Proposition 5.3 and Theorem 6.1]. Then,

$$\beta_0^F(R) = e_{HK}(\mathfrak{m}, R) = \lim_{e \rightarrow \infty} \frac{\beta_0({}^eR)}{q^{(d+\alpha)}} = \lim_{e \rightarrow \infty} \frac{a_e}{q^{(d+\alpha)}} + \lim_{e \rightarrow \infty} \frac{\beta_0(M_e)}{q^{(d+\alpha)}} = s(R) + \lim_{e \rightarrow \infty} \frac{\beta_0(M_e)}{q^{(d+\alpha)}}.$$

Hence,

$$\beta_i^F(R) = \lim_{e \rightarrow \infty} \frac{\beta_i({}^eR)}{q^{(d+\alpha)}} = \lim_{e \rightarrow \infty} \frac{\beta_i(M_e)}{q^{(d+\alpha)}} = \lim_{e \rightarrow \infty} \frac{\beta_0(M_e)}{q^{(d+\alpha)}} = e_{HK}(\mathfrak{m}, R) - s(R)$$

for  $i > 0$ .

As for the Hilbert-Kunz multiplicity, the Frobenius Betti numbers also increase after taking the quotient by a nonzero divisor.

**Proposition 3.3.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ ,  $M$  be an  $R$ -module of finite length, and  $x \in \mathrm{ann}(M)$  be a nonzero divisor on  $R$ . Then

$$\beta_{i,R}^F(M, R) = \lim_{e \rightarrow \infty} \frac{\lambda(\mathrm{Tor}_i^R(M, {}^eR))}{q^{(d+\alpha)}} \leq \beta_{i,R/(x)}^F(N, R/(x)) = \lim_{e \rightarrow \infty} \frac{\lambda(\mathrm{Tor}_i^{R/(x)}(M, {}^e(R/(x))))}{q^{(d-1+\alpha)}},$$

where the subscripts indicate over which ring we are computing the Frobenius Betti numbers. In particular,  $\beta_{i,R}^F(R) \leq \beta_{i,R/(x)}^F(R/(x))$ .

*Proof.* Let  $G_\bullet \rightarrow {}^eR$  be a minimal free resolution of  ${}^eR$ . Let  $\overline{R}$  denote  $R/xR$ . We have that  $\overline{G}_\bullet = G_\bullet \otimes_R \overline{R}$  is a free resolution for  ${}^eR \otimes_R \overline{R}$  as an  $\overline{R}$ -module. Furthermore, we have that  $H_0(\overline{G}_\bullet) = {}^eR \otimes_R \overline{R}$ . This is a consequence of the fact that  $H_i(\overline{G}_\bullet) = \mathrm{Tor}_i^R({}^eR, \overline{R}) = 0$  for  $i > 0$  because  $x$  is a nonzero divisor on  $R$  and  ${}^eR$ .

Since  $x \in \text{ann}(M)$ , we have that

$$\text{Tor}_i^R(M, {}^eR) = H_i(M \otimes_R G_\bullet) = H_i(M \otimes_{\overline{R}} \overline{R} \otimes_R G_\bullet) = H_i(M \otimes_{\overline{R}} \overline{G}_\bullet) = \text{Tor}_i^{\overline{R}}(M, {}^eR \otimes_R \overline{R}).$$

Since  $x$  is a nonzero divisor on  $R$ , there is a filtration

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_{q^\alpha} = {}^eR \otimes_R \overline{R}$$

such that  $L_{r+1}/L_r = {}^e(\overline{R})$ . As a consequence,  $\lambda(\text{Tor}_i^{\overline{R}}(M, {}^eR \otimes_R \overline{R})) \leq q^\alpha \lambda(\text{Tor}_i^{\overline{R}}(M, {}^e\overline{R}))$

Then

$$\lim_{e \rightarrow \infty} \frac{\lambda(\text{Tor}_i^R(M, {}^eR))}{q^{(d+\alpha)}} \leq \lim_{e \rightarrow \infty} \frac{\lambda(\text{Tor}_i^{\overline{R}}(M, {}^e\overline{R}))}{q^{(d-1+\alpha)}}$$

□

We now introduce  $\mu_i^F(M, N)$ , a dual version of  $\beta_i^F(M, N)$ , which is defined in terms of  $\text{Ext}$ . In Proposition 3.11, we establish a relation between these asymptotic invariants.

**Definition 3.4.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ , let  $M$  be an  $R$ -module of finite length, and let  $N$  be a finitely generated  $R$ -module. We define

$$\mu_i^F(M, N) = \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^eN))}{q^{(d+\alpha)}},$$

Next, we prove that the numbers  $\mu_i^F(M, N)$  are well defined. The proof is essentially the same as the one for  $\beta_i^F(M, N)$ , as it uses the main result in [Sei89]. Nonetheless, we include it here for completeness.

**Proposition 3.5.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ , let  $M$  be an  $R$ -module of finite length, and let  $N$  be a finitely generated  $R$ -module. Then,  $\lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(N, {}^eM))}{q^{(d+\alpha)}}$  exists. Moreover, if  $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$  is a short exact sequence, then

$$\lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^eN_2))}{q^{(d+\alpha)}} = \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^eN_1))}{q^{(d+\alpha)}} + \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^eN_3))}{q^{(d+\alpha)}}.$$

*Proof.* Let  $G_\bullet \rightarrow M$  be a minimal free resolution of  $M$  and define

$$g_e(N) = \lambda(H^i(\text{Hom}_R(G_\bullet, {}^eN))).$$

Let  $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$  be a short exact sequence of finitely generated  $R$ -modules. We have that  $g_e(N_2) \leq g_e(N_1) + g_e(N_3)$  and equality holds if the sequence splits. Then,

$$\lim_{e \rightarrow \infty} \frac{g_e(N)}{q^{(d+\alpha)}} = \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^eN))}{q^{(d+\alpha)}}$$

exists, and it is additive in short exact sequences [Sei89]. □

**Proposition 3.6.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ ,  $M$  be an  $R$ -module of finite length, and  $N$  be a finitely generated  $R$ -module. Let  $\Lambda$  be the set of all prime ideals  $\mathfrak{p}$  such that  $\dim R/\mathfrak{p} = \dim R$ . We have that

$$\beta_i^F(M, N) = \sum_{\mathfrak{p} \in \Lambda} \beta_i^F(M, R/\mathfrak{p}) \lambda_{R/\mathfrak{p}}(N_{\mathfrak{p}})$$

and

$$\mu_i^F(M, N) = \sum_{\mathfrak{p} \in \Lambda} \mu_i^F(M, R/\mathfrak{p}) \lambda_{R/\mathfrak{p}}(N_{\mathfrak{p}}).$$



*Proof.* We only prove the first statement, since the proof of the second is completely analogous. Let  $0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_h = N$  be a filtration for  $N$  such that  $N_j/N_{j-1} \cong R/\mathfrak{p}_j$ , where  $\mathfrak{p}_j \subseteq R$  is a prime ideal. Thus, we have short exact sequences  $0 \rightarrow N_{j-1} \rightarrow N_j \rightarrow R/\mathfrak{p}_j \rightarrow 0$ . We deduce that  $\beta_i^F(M, N) = \sum_{j=1}^h \beta_i^F(M, R/\mathfrak{p}_j)$  [Sei89, Proposition 1(b)]. In addition, we have that  $\beta_i^F(M, R/\mathfrak{p}_j) = 0$  whenever  $\dim(R/\mathfrak{p}_j) < \dim(R)$  [Sei89, Proposition 1(a)]. To prove the result, we need to count the number of times that a prime  $\mathfrak{p}$  such that  $\dim R/\mathfrak{p} = \dim R$  appears in the prime filtration. This number is obtained by localizing the above filtration at  $\mathfrak{p}$ , and counting the length of the resulting chain. Since the localized chain is a composition series of the module  $N_{\mathfrak{p}}$ , we obtain that the number of times  $\mathfrak{p}$  appears in the prime filtration above is given by  $\lambda_{R_{\mathfrak{p}}}(N_{\mathfrak{p}})$ . Then

$$\beta_i^F(M, N) = \sum_{\substack{j=1 \\ \mathfrak{p}_j \in \Lambda}}^h \beta_i^F(R/\mathfrak{p}_j) = \sum_{\mathfrak{p} \in \Lambda} \beta_i^F(M, R/\mathfrak{p}) \lambda_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}).$$

□

**Remark 3.7.** It follows from Proposition 3.6 that, if  $\beta_i^F(M, R) = 0$  for some  $i \in \mathbb{N}$ , we have that  $\beta_i^F(N, R/\mathfrak{p}) = 0$  for every minimal prime of  $R$  such that  $\dim(R/\mathfrak{p}) = d$ . Therefore, if this is the case,  $\beta_i^F(M, N) = 0$  for every finitely generated  $R$ -module  $N$ , using again Proposition 3.6. A similar statement holds for  $\mu_i^F(M, R)$ .

The following theorem is related to results of Chang [Cha97, Lemma 1.20 and Corollary 2.4], and in some cases it follows from them. We present a different proof that does not use spectral sequences.

**Theorem 3.8.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ ,  $M$  be an  $R$ -module of finite length, and  $N$  be a finitely generated  $R$ -module. Then

$$\lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^e N))}{q^{(i+1+\alpha)}} = 0$$

for  $i < d$ . In particular,  $\mu_i^F(M, N) = 0$  for  $i < d$ .

*Proof.* Our proof will be by induction on  $n = \dim(N)$ .

If  $n = 0$ , we have that  $h = \lambda(N)$  is finite. There is a filtration

$$0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_h = N$$

such that  $N_j/N_{j-1} \cong K$ . From the short exact sequences  $0 \rightarrow N_{j-1} \rightarrow N_j \rightarrow K \rightarrow 0$ , we have that

$$\lambda(\text{Ext}_R^i(M, {}^e N_j)) \leq \lambda(\text{Ext}_R^i(K, {}^e N_{j-1})) + \lambda(\text{Ext}_R^i(K, {}^e K)).$$

Since

$$\lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^e K))}{q^{(i+1+\alpha)}} = \lim_{e \rightarrow \infty} \frac{q^\alpha \lambda(\text{Ext}_R^i(M, K))}{q^{(i+1+\alpha)}} = \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, K))}{q^{(i+1)}} = 0,$$

we have that

$$\lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^e N))}{q^{(i+1+\alpha)}} = 0$$

by an inductive argument.

Suppose that our claim is true for modules of dimension less or equal to  $n - 1$ . There is a filtration

$$0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_h = N$$

such that  $N_j/N_{j-1} \cong R/\mathfrak{p}_j$ , where  $\mathfrak{p}_j \subset R$  is a prime ideal. From the short exact sequences  $0 \rightarrow N_{j-1} \rightarrow N_j \rightarrow R/\mathfrak{p}_j \rightarrow 0$  we have that

$$\lambda(\text{Ext}_R^i(M, {}^e N_j)) \leq \lambda(\text{Ext}_R^i(M, {}^e N_{j-1})) + \lambda(\text{Ext}_R^i(M, {}^e(R/\mathfrak{p}_j))).$$

It suffices to show that

$$(3.0.1) \quad \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^e(R/\mathfrak{p}_j)))}{q^{(i+1+\alpha)}} = 0$$

for primes  $\mathfrak{p}_j$  such that  $\dim_R(R/\mathfrak{p}_j) = n = \dim_R N$ . Let  $T = R/\mathfrak{p}_j$ . Let  $x \in \text{Ann}_R M \setminus \mathfrak{p}_j$ , which exists because  $\dim_R T = \dim_R N > 0 = \dim_R M$ . We have a short exact sequence

$$0 \longrightarrow {}^e T \xrightarrow{x} {}^e T \longrightarrow {}^e T/x({}^e T) \longrightarrow 0,$$

which induces a long exact sequence

$$(3.0.2) \quad \dots \rightarrow \text{Ext}_R^i(M, {}^e T) \xrightarrow{0} \text{Ext}_R^i(M, {}^e T) \longrightarrow \text{Ext}_R^i(M, {}^e T/x({}^e T)) \rightarrow \dots$$

Then, for every  $i$ ,

$$\lambda(\text{Ext}_R^i(M, {}^e T)) \leq \lambda(\text{Ext}_R^{i-1}(M, {}^e T/x({}^e T))).$$

We have a filtration

$$0 = L_0 \subseteq L_1 \subseteq \dots \subseteq L_{q^\alpha} = {}^e T/x({}^e T)$$

such that  $L_{r+1}/L_r = {}^e(T/xT)$  because  $x$  is not a zero divisor of  $T$ . From the induced long exact sequence by  $\text{Ext}_R^i(M, -)$ , we have that

$$\lambda(\text{Ext}_R^i(M, {}^e T/x({}^e T))) \leq q^\alpha \lambda(\text{Ext}_R^i(M, {}^e(T/xT))).$$

Therefore

$$\begin{aligned} \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^e T))}{q^{(i+1+\alpha)}} &\leq \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^{i-1}(M, {}^e T/x({}^e T)))}{q^{(i+1+\alpha)}} \\ &\leq \lim_{e \rightarrow \infty} \frac{q^\alpha \lambda(\text{Ext}_R^{i-1}(M, {}^e(T/xT)))}{q^{(i+1+\alpha)}} \\ &= \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^{i-1}(K, {}^e(T/xT)))}{q^{(i+\alpha)}} \\ &= 0 \text{ because } \dim T/xT = n - 1. \end{aligned}$$

□

**Corollary 3.9.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ . Let  $N$  be a finitely generated  $R$ -module, and let  $C$  be an  $R$ -module such that, for all  $e \gg 0$ ,  $C^{\theta_e}$  is a direct summand of  ${}^e N$  for some  $\theta_e \in \mathbb{N}$ . Assume that  $\theta = \limsup_{e \rightarrow \infty} \frac{\theta_e}{q^{(d+\alpha)}} > 0$ . Then, for all  $R$ -modules  $M$  of finite length, and all integers  $i$ , we have

$$\mu_i^F(M, N) \geq \theta \cdot \lambda(\text{Ext}_R^i(M, C)).$$

In particular,  $C$  is a maximal Cohen-Macaulay module.



*Proof.* We have

$$\mu_i^F(M, N) = \lim_{e \rightarrow \infty} \frac{\lambda(\text{Ext}_R^i(M, {}^eN))}{q^{(d+\alpha)}} \geq \limsup_{e \rightarrow \infty} \frac{\theta_e \cdot \lambda(\text{Ext}_R^i(M, C))}{q^{(d+\alpha)}} = \theta \cdot \lambda(\text{Ext}_R^i(M, C)).$$

Using  $M = K$  in Theorem 3.8, we obtain that  $\mu_i^F(K, N) = 0$  for all  $i < d$ . It follows from the inequality that  $\text{Ext}_R^i(K, C) = 0$  for all  $i < d$ , and then  $C$  is a maximal Cohen-Macaulay module.  $\square$

**Remark 3.10.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ , and  $N$  be a finitely generated  $R$ -module. We say that an  $R$ -module  $C$  is an  $F$ -contributor of  $N$  if  $C^{\theta_e}$  is a direct summand of  ${}^eN$  for  $e \gg 0$ , and  $\limsup_{e \rightarrow \infty} \frac{\theta_e}{q^{(d+\alpha)}} > 0$  [Yao05]. Corollary 3.9 shows that every  $F$ -contributor is a maximal Cohen-Macaulay module. This was already noted by Yao [Yao05, Lemma 2.2] when  $N$  has finite  $F$ -representation type.

The following proposition shows that taking limits with respect to Tor or Ext give the same invariants up to a shift in the homological degrees.

**Proposition 3.11.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ , and  $M$  be an  $R$ -module of finite length. Then

$$\beta_i^F(M, R) = \mu_{d+i}^F(M, R)$$

for every  $i \in \mathbb{N}$ .

*Proof.* Since  $\beta_i^F(M, R)$  and  $\mu_{d+i}^F(M, R)$  are not affected by completion at  $\mathfrak{m}$ , we may assume that  $R$  is a complete local ring. In this case,  $R$  has a dualizing complex  $D_R^\bullet$  by Remark 2.2. We have that

$$\beta_i^F(M, R) = \mu_{d+i}^F(M, H^0(D_R^\bullet))$$

by [Cha97, Proposition 2.3(2)]. Let  $\Lambda$  be the set of all prime ideals of  $R$  such that  $\dim R/\mathfrak{p} = \dim R$ . Let  $\mathfrak{p} \in \Lambda$ . We have that  $(H^0(D_R^\bullet))_{\mathfrak{p}} = H^0(D_{R_{\mathfrak{p}}}^\bullet) = \omega_{R_{\mathfrak{p}}}$  by Remark 2.2. We have that  $\omega_{R_{\mathfrak{p}}} = \text{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}, E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}))$  and  $\lambda_{R_{\mathfrak{p}}}(\omega_{R_{\mathfrak{p}}}) = \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ . Finally, by Proposition 3.6

$$\begin{aligned} \mu_{d+i}^F(M, H^0(D_R^\bullet)) &= \sum_{\mathfrak{p} \in \Lambda} \mu_{d+i}^F(M, R/\mathfrak{p}) \lambda_{R_{\mathfrak{p}}}(H^0(D_{R_{\mathfrak{p}}}^\bullet)) \\ &= \sum_{\mathfrak{p} \in \Lambda} \mu_{d+i}^F(M, R/\mathfrak{p}) \lambda_{R_{\mathfrak{p}}}(\omega_{R_{\mathfrak{p}}}) \\ &= \sum_{\mathfrak{p} \in \Lambda} \mu_{d+i}^F(M, R/\mathfrak{p}) \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \\ &= \mu_{d+i}^F(M, R). \end{aligned}$$

$\square$

**Remark 3.12.** If  $R$  itself has an  $F$ -contributor  $C$ , then we get a relation involving the  $\beta_i^F$ 's. In fact, by Proposition 3.11, we have  $\beta_i^F(M, R) = \mu_{d+i}^F(M, R)$  for all  $i \in \mathbb{N}$ . Thus, in the notation of Corollary 3.9, we have  $\beta_i^F(M, R) \geq \theta \cdot \lambda(\text{Ext}_R^{d+i}(M, C))$ .

We end this section with a proposition that shows how  $\beta_i^F(M, N)$  behaves under some flat ring extensions. First, we need a different way to compute  $\beta_i^F(M, N)$ .

**Remark 3.13.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ , let  $M$  be an  $R$ -module of finite length, and let  $N$  be a finitely generated  $R$ -module. Let  $G_\bullet = (G_j, \varphi_j)_{j \geq 0}$  denote a minimal free resolution of  $M$ . Let  $G_\bullet^{[q]}$  be the complex  $(G_j, \varphi_j^{[q]})_{j \geq 0}$ , where the matrix of  $\varphi_j^{[q]}$  has as entries the  $q$ -th powers of the entries in the matrix of  $\varphi_j$ . We have that

$$\lambda(\mathrm{Tor}_i^R(M, {}^e N)) = q^\alpha \lambda(H_i(G_\bullet^{[q]} \otimes_R N)).$$

Hence,

$$\beta_i^F(M, N) = \lim_{q \rightarrow \infty} \frac{\lambda(H_i(G_\bullet^{[q]} \otimes_R N))}{q^d}.$$

**Proposition 3.14.** Let  $(R, \mathfrak{m}, K) \rightarrow (S, \mathfrak{n}, L)$  be a flat extension of two  $F$ -finite local rings of characteristic  $p > 0$ . Let  $M$  be a finite length  $R$ -module. Let  $\alpha = \log_p[K : K^p]$  and  $\theta = \log_p[L : L^p]$ . Suppose that  $\mathfrak{m}S = \mathfrak{n}$ . Then,

$$\beta_{i,R}^F(M, R) = \lim_{e \rightarrow \infty} \frac{\lambda(\mathrm{Tor}_i^R(M, {}^e R))}{p^{e(d+\alpha)}} = \lim_{e \rightarrow \infty} \frac{\lambda(\mathrm{Tor}_i^S(M \otimes_R S, {}^e S))}{p^{e(d+\theta)}} = \beta_{i,S}^F(M \otimes_R S, S).$$

In particular, we have that  $\beta_{i,R}^F(M, R) = \beta_{i,\widehat{R}}^F(\widehat{M}, \widehat{R})$ .

*Proof.* Let  $q = p^e$ . We have that

$$\begin{aligned} \frac{\lambda_R(\mathrm{Tor}_i^R(M, {}^e R))}{q^\alpha} &= \lambda_R(H_i(G_\bullet^{[q]})) \text{ by Remark 3.13.} \\ &= \lambda_S(H_i(G_\bullet^{[q]} \otimes_R S)) \text{ because } S \text{ is flat and } \mathfrak{m}S = \mathfrak{n}. \\ &= \lambda_S(H_i((G_\bullet \otimes_R S)^{[q]})) \text{ because } G_\bullet \text{ is free.} \\ &= \frac{\lambda_S(\mathrm{Tor}_i^S(M \otimes_R S, {}^e S))}{q^\theta} \text{ by Remark 3.13 and because } S \text{ is flat.} \end{aligned}$$

After dividing by  $q^d$  and taking limits, we have that

$$\beta_{i,R}^F(M, R) = \beta_{i,S}^F(M \otimes_R S, S).$$

□

#### 4. RELATIONS WITH PROJECTIVE DIMENSION

Let  $(R, \mathfrak{m}, K)$  be a local  $F$ -finite ring of characteristic  $p > 0$ , and let  $M$  be an  $R$ -module of finite length. In this section we investigate when the vanishing of  $\beta_i^F(M, R)$  detects whether  $M$  has finite projective dimension.

We first recall known results in this direction. We have that  $R$  is a regular ring if and only if  $\beta_i^F(R) = \beta_i^F(K, R) = 0$  for some  $i \geq 1$  [AL08, Corollary 3.2]. Let  $M$  be a finitely generated  $R$ -module. If  $M$  has finite projective dimension, then  $\mathrm{Tor}_i^R(M, {}^e R) = 0$  for all  $i > 0$  and all  $e \geq 0$  [PS73, Théorème 1.7]. Conversely, if  $\mathrm{Tor}_i^R(M, {}^e R) = 0$  for infinitely many  $e$  and all  $i > 0$ , then  $M$  has finite projective dimension [Her74, Theorem 3.1]. In fact, even more is true: if  $\mathrm{Tor}_i^R(M, {}^e R) = 0$  for  $\mathrm{depth}(R) + 1$  consecutive values of  $i$  and some  $e \gg 0$ , then  $M$  has finite projective dimension [KL98, Proposition 2.6] (see also [Mil03, Theorem 2.2.8]). Now, suppose that  $R$  is a complete intersection. If  $\beta_i^F(M, R) = 0$  for some  $i > 0$ , then  $M$  has finite projective dimension by [Mil00, Corollary 2.5] (see also [DS13, Corollary 4.11]).

**Proposition 4.1.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ , and let  $M$  be an  $R$ -module of finite length. Suppose that there is a regular local ring  $(A, \mathfrak{n}, L)$  and a map of local rings  $\phi : R \rightarrow A$  such that  $A$  is finitely generated as an  $R$ -module, and  $\dim A = d$ . If

$$\beta_j^F(M, R) = \beta_{j+1}^F(M, R) = \dots = \beta_{j+d}^F(M, R) = 0,$$

then  $M$  has finite projective dimension.

*Proof.* We note that  $\log_p[L : L^p] = \log_p[K : K^p] = \alpha < \infty$ , and so,  $A$  is  $F$ -finite. Since  $A$  is regular and local,  ${}^e A \cong \bigoplus^{q^{(d+\alpha)}} A$ . Let  $x_1, \dots, x_d \in A$  be a set of generators for  $\mathfrak{n}$ , and let  $I_r := (x_1, \dots, x_r)A$ . By induction on  $r$  we will show that

$$(4.0.3) \quad \mathrm{Tor}_{j+r}^R(M, A/I_r) = \dots = \mathrm{Tor}_{j+d}^R(M, A/I_r) = 0$$

for every  $r$ . If  $r = 0$ , we have that  $\mathrm{Tor}_i^R(M, {}^e A) = \bigoplus^{q^{(d+\alpha)}} \mathrm{Tor}_i^R(M, A)$  for every  $i \in \mathbb{N}$ . Then,  $\lambda(\mathrm{Tor}_i^R(M, {}^e A)) = q^{(d+\alpha)} \lambda(\mathrm{Tor}_i^R(M, A))$ , and thus

$$\beta_i^F(M, A) = \lambda(\mathrm{Tor}_i^R(M, A)).$$

Since  $A$  is finitely generated, and since  $\beta_i^F(M, R) = 0$  for  $i = j, \dots, j+d$  by assumption, we have that  $\beta_j^F(M, A) = \dots = \beta_{j+d}^F(M, A) = 0$  by Remark 3.7. Hence,  $\mathrm{Tor}_j^R(M, A) = \dots = \mathrm{Tor}_{j+d}^R(M, A) = 0$ . We suppose that (4.0.3) holds for  $r-1$  and prove it for  $r$ . We have a short exact sequence

$$0 \longrightarrow A/I_{r-1} \xrightarrow{x_r} A/I_{r-1} \longrightarrow A/I_r \longrightarrow 0.$$

This induces a long exact sequence

$$\dots \longrightarrow \mathrm{Tor}_i^R(M, A/I_{r-1}) \xrightarrow{x_r} \mathrm{Tor}_i^R(M, A/I_{r-1}) \longrightarrow \mathrm{Tor}_i^R(M, A/I_r) \longrightarrow \mathrm{Tor}_{i-1}^R(M, A/I_{r-1}) \longrightarrow \dots$$

Since  $\mathrm{Tor}_{j+r-1}^R(M, A/I_{r-1}) = \dots = \mathrm{Tor}_{j+d}^R(M, A/I_{r-1}) = 0$ , we have that  $\mathrm{Tor}_{j+r}^R(M, A/I_r) = \dots = \mathrm{Tor}_{j+d}^R(M, A/I_r) = 0$ , proving the claim. In particular, we get  $\mathrm{Tor}_{j+d}^R(M, A/I_d) = 0$ . Since  $L = A/I_d$  is a finite field extension of  $K$ , we have

$$0 = \lambda(\mathrm{Tor}_{j+d}^R(M, A/I_d)) = [L : K] \cdot \lambda(\mathrm{Tor}_{j+d}^R(M, K)).$$

Therefore,  $\mathrm{Tor}_{j+d}^R(M, K) = 0$  and  $M$  has finite projective dimension.  $\square$

**Lemma 4.2.** Let  $(R, \mathfrak{m}, K)$  be a local ring of characteristic  $p > 0$ . Suppose that there is an  $R$ -module  $N$  of dimension  $d$  that has an  $F$ -contributor  $C$ . Let  $M$  be an  $R$ -module of finite length. If  $\beta_i^F(M, N) = 0$ , then  $\mathrm{Tor}_i^R(M, {}^e C) = 0$  for every  $e \geq 0$ . In particular, if  $R$  is strongly  $F$ -regular of positive dimension  $d$ , and  $\beta_i^F(M, R) = 0$  for  $d$  consecutive values of  $i$ , then  $M$  has finite projective dimension.

*Proof.* For  $e' \gg 0$  and  $q' = p^{e'}$ , we have that  $C^{\theta_{e'}}$  is a direct summand of  ${}^{e'} N$ , for some  $\theta_{e'} \in \mathbb{N}$  such that  $\limsup \frac{\theta_{e'}}{q^{(d+\alpha)}} > 0$ . We note that  $({}^e C)^{\theta_{e'}}$  is a direct summand of  ${}^{e+e'} N$  for all  $e \geq 0$ . Then

$$\left( \limsup_{e' \rightarrow \infty} \frac{\theta_{e'}}{q^{(d+\alpha)}} \right) \frac{\lambda(\mathrm{Tor}_i^R(M, {}^e C))}{q^{(d+\alpha)}} \leq \lim_{e' \rightarrow \infty} \frac{\lambda(\mathrm{Tor}_i^R(M, {}^{e+e'} N))}{qq^{(d+\alpha)}} = \beta_i^F(M, N) = 0.$$

It follows that  $\mathrm{Tor}_i^R(M, {}^e C) = 0$ . If  $R$  is strongly  $F$ -regular, then  $R$  is an  $F$ -contributor of itself. In addition,  $R$  is Cohen-Macaulay, and if  $\mathrm{Tor}_i^R(M, {}^e R) = 0$  for  $d$  consecutive values of  $i$  and for  $e \gg 0$ , we have that  $M$  has finite projective dimension [KL98, Proposition 2.6] (see also [Mil03, Theorem 2.2.11]).  $\square$

**Proposition 4.3.** [CHKV06, Corollary 3.3] Let  $(R, \mathfrak{m}, K)$  be a local ring, let  $I$  be an integrally closed  $\mathfrak{m}$ -primary ideal, and let  $N$  be a finitely generated  $R$ -module. Then  $\mathrm{pd}_R(N) < i$  if and only if  $\mathrm{Tor}_i^R(N, R/I) = 0$ .

In particular, Proposition 4.3 shows that, if  $\mathrm{Tor}_i^R(R/I, {}^eR) = 0$  for some  $e \geq 1$ , then  $R$  is regular [Kun69, Theorem 2.1]. We now present a similar result for Frobenius Betti numbers.

**Proposition 4.4.** Let  $(R, \mathfrak{m}, K)$  be a reduced local ring of characteristic  $p > 0$ . Suppose that there exists an  $R$ -module  $N$  of dimension  $d$  that has an  $F$ -contributor  $C$ . If  $I$  is an integrally closed  $\mathfrak{m}$ -primary ideal such that  $\beta_i^F(R/I, N) = 0$  for some  $i > 0$ , then  $R$  is regular.

*Proof.* By Lemma 4.2, we have that  $\mathrm{Tor}_i(R/I, {}^eC) = 0$  for every  $e \geq 0$ , and thus  ${}^eC$  has finite projective dimension by [CHKV06, Corollary 3.3]. Since  ${}^eC$  is a maximal Cohen-Macaulay module [Yao05, Lemma 2.2] (see Remark 3.10), we have that  ${}^eC$  is a free module for every  $e \geq 0$ . In particular,  ${}^1C \cong \bigoplus_{\alpha} R$  and  ${}^2C = {}^1(\bigoplus_{\alpha} R) \cong \bigoplus_{\alpha} {}^1R$  is free as well. Therefore,  ${}^1R$  is free and  $R$  is regular [Kun69, Theorem 2.1].  $\square$

We now focus on one-dimensional rings. In this case, we can find a characterization of the vanishing of  $\beta_i^F(M, R)$ . We first prove two lemmas.

**Lemma 4.5.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional complete local domain of characteristic  $p > 0$ , with  $K$  algebraically closed. Then there exists a parameter  $x \in R$  such that  $(x^q) = \mathfrak{m}^{[q]}$  for all  $q = p^e \gg 0$ . Furthermore, if  $V$  denotes the integral closure of  $R$  in its field of fractions, then  ${}^eR \cong \bigoplus V$  for all  $e \gg 0$  (as  $R$ -modules).

*Proof.* Since  $R$  is a complete domain, we have that  $(V, \mathfrak{m}_V, K)$  is a one dimensional, integrally closed, local domain. Hence,  $V$  is a DVR. Let  $x \in R$  be a minimal reduction of  $\mathfrak{m}$ , and let  $v$  denote the order valuation on  $V$ . Let  $x, y_1, \dots, y_n$  be a minimal generating set of the maximal ideal. We claim that we can choose the elements  $y_i$ 's such that  $v(x) < v(y_i)$  for all  $i = 1, \dots, n$ . We have  $v(x) \leq v(y_i)$  for all  $i$  because  $x$  is a minimal reduction of  $\mathfrak{m}$  [SH06, Proposition 6.8.1]. If equality holds, say for  $i = 1$ , we have that  $y_1/x = \alpha \in K_V = K$  since  $K$  is algebraically closed. Fix a lifting  $u \in R$  of  $\alpha$ . If we replace  $y_1$  for  $y_1' := y_1 - ux$  we have that  $x, y_1', \dots, y_n$  is still a minimal generating set of  $\mathfrak{m}$ . Now  $v(x) < v(y_1')$ , since  $y_1'/x \in \mathfrak{m}_V$ . Similarly, if necessary, we may replace each  $y_i$  to obtain our claim. Since the conductor  $C$  is  $\mathfrak{m}_V$ -primary, for all  $e \gg 0$  and all  $i = 1, \dots, n$ , we have that  $(y_i/x)^q = r_i \in \mathfrak{m}_V^{[q]} \subseteq C \subseteq R$ . Thus  $y_i^q = r_i x^q \in (x)^q$ . This shows the first part of the lemma.

We now focus on the second part of the lemma. Since  $K$  is algebraically closed,  $R$  and  $K$  have the same residue field. It then follows that  $R \subseteq V = R + \mathfrak{m}_V$ . Since  $R$  is a domain, we can identify  ${}^eR$  with  $R^{1/q}$ , the ring of  $q$ -th roots of  $R$ . For  $w \in V$ , we can write  $w = u + v$ , for some  $u \in R$  and  $v \in \mathfrak{m}_V$ . Therefore, we have that  $\mathfrak{m}_V^{[q]} \subseteq C \subseteq R$  for  $e \gg 0$ , because  $C$  is  $\mathfrak{m}_V$  primary. This shows that  $w^q = u^q + v^q \in R$ , that is  $w \in R^{1/q}$ . Thus, for  $e \gg 0$ , we have  $R \subseteq V \subseteq {}^eR$ . Hence,  ${}^eR$  is a  $V$ -module. Since  $V$  is a DVR,  ${}^eR$  decomposes into a  $V$ -free part and a  $V$ -torsion part. Therefore  ${}^eR$  is torsion free as a  $V$ -module because  $R$  is a domain. Thus,  ${}^eR \cong \bigoplus_q V$ . Finally, the  $V$ -module structure on  ${}^eR$  is compatible with the inclusion  $R \subseteq V$ ; therefore,  ${}^eR \cong \bigoplus V$  is also an isomorphism of  $R$ -modules.  $\square$

**Lemma 4.6.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional local ring of characteristic  $p > 0$ . Let  $(G_j, \varphi_j)_{j \geq 0}$  be a minimal free resolution of a finite length  $R$ -module  $M$ . Suppose there exists

$i \geq 0$  such that  $\text{Im}(\varphi_{i+1}) \not\subseteq \mathfrak{p}G_i$  for some  $\mathfrak{p} \in \text{Min}(R)$ . Then

$$\beta_i^F(M, R) = \lim_{e \rightarrow \infty} \frac{\lambda(\text{Tor}_i^R(M, {}^eR))}{q^e} > 0.$$

*Proof.* By the Cohen Structure Theorem, we have that  $\widehat{R} = K[[x_1, \dots, x_n]]/I$  for some  $n \in \mathbb{N}$  and some ideal  $I \subseteq K[[x_1, \dots, x_n]]$ . Let  $S = L[[1, \dots, x_n]]/I'$ , where  $L$  is the algebraic closure of  $K$  and  $I' = I \cap L[[1, \dots, x_n]]$ . Every inclusion  $K \rightarrow L$  gives a flat extension  $R \rightarrow S$  such that  $\mathfrak{m}S$  is the maximal ideal of  $S$ . If  $\text{Im}(\varphi_{i+1} \otimes_R 1_S) = \text{Im}(\varphi_{i+1}) \otimes_R S$  is contained in a minimal prime of  $S$ , then  $\text{Im}(\varphi_{i+1})$  must be contained in the contraction of such minimal prime to  $R$ . Then, we can assume that  $R$  is complete and that  $K$  is algebraically closed by Proposition 3.14.

Let  $\overline{R}$  denote  $R/\mathfrak{p}$ ,  $\overline{x}$  the class of the element  $x$  modulo  $\mathfrak{p}$ , and  $V$  the integral closure of  $\overline{R}$ . Since  $R/\mathfrak{p}$  is a one-dimensional complete local domain, by Lemma 4.5 we can choose  $0 \neq \overline{x} \in \overline{R}$  a minimal reduction of  $\overline{\mathfrak{m}} := \mathfrak{m}/\mathfrak{p}$  and  $q_0 = p^{e_0}$  such that  $\overline{\mathfrak{m}}^{[q]} = (\overline{x}^q)$  for  $q \geq q_0$ . We may also choose  $q_0$  large enough so that  $\overline{x}^q V \cap \overline{R} \subseteq \overline{x}R$ , by using the Artin-Rees Lemma and the fact that the conductor from  $\overline{R}$  to  $V$  is primary to the maximal ideal. In particular,  $(\overline{x}^q V :_V \overline{r}) \subseteq \mathfrak{m}_V$  for every  $\overline{r} \in \overline{R}$  such that  $\overline{r} \notin \overline{x}R$ , where  $\mathfrak{m}_V$  is the maximal ideal of  $V$ , which is a DVR.

Fix  $q \geq q_0$  and consider the matrix associated to  $\overline{\varphi}_{i+1}^{[q]} := \varphi_{i+1}^{[q]} \otimes 1_{\overline{R}}$ . Since  $q \geq q_0$ ,  $\text{Im}(\varphi_{i+1}^{[q]} \otimes 1_{\overline{R}}) \subseteq \overline{\mathfrak{m}}^{[q]}G_i = (\overline{x}^q)G_i$ . Because  $\text{Im}(\varphi_{i+1}) \not\subseteq \mathfrak{p}G_i$ , by changing the basis for  $G_{i+1}$  if needed, we can assume that the matrix

$$\overline{\varphi}_{i+1}^{[q]} = \overline{x}^{q+j} \left[ \begin{array}{c|ccc} \overline{r}_1 & * & \dots & * \\ \hline \overline{r}_2 & * & \dots & * \\ \hline \vdots & \vdots & & \vdots \\ \hline \overline{r}_n & * & \dots & * \end{array} \right],$$

where we have factored out the biggest possible power of  $\overline{x}$ , so that  $\overline{r}_1 \notin (\overline{x})$ . Here  $n = \text{rk}(G_i)$ .

Let  $q' = p^{e'}$ , and consider the matrix associated to  $\overline{\varphi}_{i+1}^{[qq']}$ :

$$(4.0.4) \quad \overline{\varphi}_{i+1}^{[qq']} = \overline{x}^{(q+j)q'} \left[ \begin{array}{c|ccc} \overline{r}_1^{q'} & * & \dots & * \\ \hline \overline{r}_2^{q'} & * & \dots & * \\ \hline \vdots & \vdots & & \vdots \\ \hline \overline{r}_n^{q'} & * & \dots & * \end{array} \right].$$

We claim that  $[\overline{r}_1^{q'}, \overline{r}_2^{q'}, \dots, \overline{r}_n^{q'}]^T \in \text{Ker}(\overline{\varphi}_i^{[qq']})$ . In fact, we have that

$$\overline{x}^{qq'+jq'} \begin{bmatrix} \overline{r}_1^{q'} \\ \overline{r}_2^{q'} \\ \vdots \\ \overline{r}_n^{q'} \end{bmatrix} \in \text{Im}(\overline{\varphi}_{i+1}^{[qq']}) \subseteq \text{Ker}(\overline{\varphi}_i^{[qq']});$$

therefore,

$$\overline{\varphi}_i^{[qq']}\left(\overline{x}^{qq'+jq'} \cdot \begin{bmatrix} \overline{r}_1^{q'} \\ \overline{r}_2^{q'} \\ \vdots \\ \overline{r}_n^{q'} \end{bmatrix}\right) = \overline{x}^{qq'+jq'} \cdot \overline{\varphi}_i^{[qq']}\left(\begin{bmatrix} \overline{r}_1^{q'} \\ \overline{r}_2^{q'} \\ \vdots \\ \overline{r}_n^{q'} \end{bmatrix}\right) = 0.$$

Since  $\overline{x}^{qq'+jq'}$  is a nonzero divisor in  $\overline{R}$ , we have

$$\overline{\varphi}_i^{[qq']}\left(\begin{bmatrix} \overline{r}_1^{q'} \\ \overline{r}_2^{q'} \\ \vdots \\ \overline{r}_n^{q'} \end{bmatrix}\right) = 0,$$

which proves the claim. Thus

$$\begin{aligned} \lambda(\mathrm{Tor}_i^R(M, {}^{e+e'}(R/\mathfrak{p}))) &\geq \lambda\left(\frac{\overline{R}[\overline{r}_1^{q'}, \dots, \overline{r}_n^{q'}]^T + \mathrm{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right)}{\mathrm{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right)}\right) \\ &\geq \lambda\left(\frac{(\overline{r}_1^{q'}) + (\overline{x}^{qq'})}{(\overline{x}^{qq'})}\right), \end{aligned}$$

because  $\mathrm{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right) \subseteq (\overline{x}^{qq'})G_i$ . This comes from the expression of  $\overline{\varphi}_{i+1}^{[qq']}$  in (4.0.4). We also have projected onto the first component of  $G_i$ . We now have a cyclic module which is isomorphic to the quotient of  $\overline{R}$  by the ideal  $(\overline{x}^{qq'} : \overline{r}_1^{q'})$ .

We claim that there exists an integer  $q_1 = p^{e_1}$  such that for all  $q'$ ,

$$(\overline{x}^{qq'} : \overline{r}_1^{q'}) \subseteq (\overline{x}^{q'/q_1}).$$

Assuming the claim and lifting back to  $R$ , we get

$$\begin{aligned} \lambda(\mathrm{Tor}_i^R(M, {}^{e+e'}\overline{R})) &\geq \lambda\left(\frac{(\overline{r}_1^{q'}) + (\overline{x}^{qq'})}{(\overline{x}^{qq'})}\right) \\ &\geq \lambda\left(\frac{\overline{R}}{(\overline{x}^{q'/q_1})}\right). \end{aligned}$$

Dividing by  $qq'$  and taking the limit as  $e' \rightarrow \infty$ , we get

$$\begin{aligned} \beta_i^F(M, \overline{R}) &= \lim_{e' \rightarrow \infty} \frac{\lambda(\mathrm{Tor}_i^R(M, {}^{e+e'}\overline{R}))}{qq'} \\ &\geq \lim_{e' \rightarrow \infty} \frac{\lambda(\overline{R}/(\overline{x}^{q'/q_1}))}{qq'} = \frac{1}{qq_1} e_{HK}(x, \overline{R}) > 0. \end{aligned}$$

Since  $\dim(R/\mathfrak{q}) = \dim(R)$  for  $\mathfrak{q} \in \mathrm{Spec}(R)$  if and only if  $\mathfrak{q} \in \mathrm{Min}(R)$ , we have

$$\beta_i^F(M, R) = \sum_{\mathfrak{q} \in \mathrm{Min}(R)} (\beta_i^F(M, R/\mathfrak{q})\lambda(R_{\mathfrak{q}})) \geq \beta_i^F(M, R/\mathfrak{p}) > 0.$$



by Proposition 3.6.

It remains to prove the claim. Suppose that  $u \in (\overline{x}^{qq'} : \overline{r}_1^{q'})$ . Then  $u \in (\overline{x}^{qq'} : \overline{r}_1^{q'})V \cap \overline{R} = (\overline{x}^q V :_V \overline{r}_1)^{[q']} \cap \overline{R} \subseteq \mathfrak{m}_V^{q'} \cap \overline{R}$  by the choice of  $q$ . Since the conductor of  $\overline{R}$  is primary to the maximal ideal it follows that there exists a  $q_1 = p^{e_1}$  such that  $\mathfrak{m}_V^{q'} \cap \overline{R} \subseteq (\overline{x}^{q'/q_1})$  as claimed.  $\square$

**Theorem 4.7.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional local ring of characteristic  $p > 0$ , and  $M$  be an  $R$ -module of finite length. Let  $(G_j, \varphi_j)_{j \geq 0}$  denote a minimal free resolution of  $M$ . Then the following are equivalent:

- (i)  $\text{Im}(\varphi_{i+1}) \subseteq H_{\mathfrak{m}}^0(G_i)$ .
- (ii)  $\text{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$  for all  $e \geq 0$ , for all  $\mathfrak{p} \in \text{Min}(R)$ .
- (iii)  $\text{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$  for all  $e \gg 0$ , for all  $\mathfrak{p} \in \text{Min}(R)$ .
- (iv)  $\beta_i^F(M, R) = 0$ .

Assume in addition that  $R$  is complete and  $K$  is algebraically closed. If  $V$  denotes the integral closure of  $R$  in its ring of fractions, then the conditions above are equivalent to

- (v)  $\text{Tor}_i^R(M, V) = 0$ .

*Proof.* We will show that (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i). We assume (i). Let  $\mathfrak{p} \in \text{Min}(R)$ . Since  $M$  has finite length we have  $M_{\mathfrak{p}} = 0$ , and thus

$$\text{Tor}_j^R(M, {}^e(R/\mathfrak{p}))_{\mathfrak{p}} = \text{Tor}_j^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, {}^e(R/\mathfrak{p})_{\mathfrak{p}}) = 0$$

for all  $j \geq 0$ . In particular, the complex

$$(G_{\bullet} \otimes {}^e(R))_{\mathfrak{p}} : \dots \longrightarrow (G_{i+1})_{\mathfrak{p}} \xrightarrow{(\varphi_{i+1}^{[q]})_{\mathfrak{p}}} (G_i)_{\mathfrak{p}} \xrightarrow{(\varphi_i^{[q]})_{\mathfrak{p}}} (G_{i-1})_{\mathfrak{p}} \longrightarrow \dots \longrightarrow (G_0)_{\mathfrak{p}} \longrightarrow 0$$

is split exact. All the entries in a matrix associated to  $\varphi_{i+1}$  are in  $H_{\mathfrak{m}}^0(R)$ , and in particular they are nilpotent. We choose  $q_0 = p^{e_0}$  such that  $\text{Im}(\varphi_{i+1}^{[q]}) = 0$  for all  $q \geq q_0$ . For such  $q$ , we have  $(\varphi_{i+1}^{[q]})_{\mathfrak{p}} \equiv 0$ ; therefore,  $(G_i)_{\mathfrak{p}}$  splits inside  $(G_{i-1})_{\mathfrak{p}}$  via  $(\varphi_i^{[q]})_{\mathfrak{p}}$ . This means that

$$(4.0.5) \quad b_i := \text{rk}((G_i)_{\mathfrak{p}}) = \text{rk}(G_i) = \text{rk}((\varphi_i^{[q]})_{\mathfrak{p}}) \quad \text{and} \quad I_{b_i}(\varphi_i^{[q]}) \not\subseteq \mathfrak{p},$$

where  $I_r(\psi)$  denotes the Fitting ideal of an homomorphism  $\psi : G \rightarrow H$  of rank  $r$  between two free modules  $G$  and  $H$ . Note that localizing and taking powers only decreases the rank of  $\varphi_i$ , and  $b_i$  is already the maximal possible rank. Thus  $b_i = \text{rk}(\varphi_i^{[q]})$  for all  $q \geq 1$ . Furthermore, if  $I_{b_i}(\varphi_i)$  was contained in  $\mathfrak{p}$ , then so would be  $I_{b_i}(\varphi_i^{[q]})$ . Hence (4.0.5) holds in fact for all  $q = p^e$ . Consider the following complex

$$0 \longrightarrow G_i \otimes R/\mathfrak{p} \xrightarrow{\varphi_i^{[q]} \otimes 1_{R/\mathfrak{p}}} G_{i-1} \otimes R/\mathfrak{p} \longrightarrow C_q \longrightarrow 0,$$

where  $C_q$  is the cokernel. By Buchsbaum-Eisenbud Theorem [BE73], the two conditions (4.0.5) ensure that it is acyclic for all  $q$ . Then,

$$\text{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = \text{Tor}_1^R(C_q, R/\mathfrak{p}) = 0.$$

for all  $e \geq 0$ . This holds for all  $\mathfrak{p} \in \text{Min}(R)$ , proving (ii).

Clearly (ii) implies (iii). We now show (iii)  $\Rightarrow$  (iv). Since for all  $\mathfrak{p} \in \text{Min}(R)$ , we have  $\text{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$  for  $e \gg 0$ , in particular  $\beta_i^F(M, R/\mathfrak{p}) = 0$ . Hence

$$\beta_i^F(M, R) = \sum_{\mathfrak{p} \in \text{Min}(R)} [\beta_i^F(M, R/\mathfrak{p}) \lambda_{R/\mathfrak{p}}(R/\mathfrak{p})] = 0.$$

We now prove (iv)  $\Rightarrow$  (i). Now suppose that  $\beta_i^F(M, R) = 0$ . By Lemma 4.6, we have

$$\text{Im}(\varphi_{i+1}) \subseteq \bigcap_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p}G_i = \sqrt{0}G_i.$$

Since the image is nilpotent, as noticed above in (4.0.5) taking  $q = 1$ , we have

$$b_i = \text{rk}(G_i) = \text{rk}(\varphi_i) \quad \text{and} \quad I_{b_i}(\varphi_i) \not\subseteq \mathfrak{p}$$

for all  $\mathfrak{p} \in \text{Min}(R)$ . Localizing the resolution at any  $\mathfrak{p} \in \text{Min}(R)$  gives a split exact complex

$$(G_\bullet)_{\mathfrak{p}} : 0 \rightarrow (G_i)_{\mathfrak{p}} \xrightarrow{(\varphi_i)_{\mathfrak{p}}} \dots \rightarrow (G_0)_{\mathfrak{p}} \rightarrow 0.$$

In particular,  $\text{Im}((\varphi_{i+1})_{\mathfrak{p}}) = (\text{Im}(\varphi_{i+1}))_{\mathfrak{p}} = 0$ . This holds for all minimal primes  $\mathfrak{p}$  of  $R$ , proving that  $\text{Im}(\varphi_{i+1}) \subseteq H_{\mathfrak{m}}^0(G_i)$ .

Finally, assume that  $R$  is complete and  $K$  is algebraically closed, and let  $V$  be integral closure of  $R$  in its ring of fractions. Let  $\mathfrak{p} \in \text{Min}(R)$  and let  $V(\mathfrak{p})$  be the integral closure of  $R/\mathfrak{p}$ , which is a DVR. By Lemma 4.5, we have that  ${}^e(R/\mathfrak{p}) \cong \bigoplus V(\mathfrak{p})$  for all  $e \gg 0$ . Condition (iii) implies that  $\text{Tor}_i^R(M, {}^e(R/\mathfrak{p})) \cong \bigoplus \text{Tor}_i^R(M, V(\mathfrak{p})) = 0$ , therefore  $\text{Tor}_i^R(M, V(\mathfrak{p})) = 0$  for all  $\mathfrak{p} \in \text{Min}(R)$ . Since  $V \cong \bigoplus_{\mathfrak{p} \in \text{Min}(R)} V(\mathfrak{p})$ , we see that (iii) implies (v). Conversely, if  $\text{Tor}_i^R(M, V) = 0$ , by the same argument, we get that  $\text{Tor}_i^R(M, V(\mathfrak{p})) = 0$  implies  $\text{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$  for all  $e \gg 0$  and for all  $\mathfrak{p} \in \text{Min}(R)$ . Then, (v) implies (iii).  $\square$

**Corollary 4.8.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional Cohen-Macaulay local ring of characteristic  $p > 0$ , and  $M$  be an  $R$ -module of finite length. Then the following are equivalent:

- (i)  $\beta_i^F(M, R) = 0$  for all  $i \geq 1$ .
- (ii)  $\beta_i^F(M, R) = 0$  for some  $i \geq 1$ .
- (iii)  $\text{pd}_R(M) < \infty$ .

*Proof.* Clearly (i) implies (ii). Now assume (ii), we want to show that (iii) holds. By assumption, there exists an integer  $i \geq 1$  such that  $\beta_i^F(M, R) = 0$ . Then, Theorem 4.7 implies that  $\text{Im}(\varphi_{i+1}) \subseteq H_{\mathfrak{m}}^0(G_i)$ , where  $(G_j, \varphi_j)_{j \geq 0}$  is a minimal free resolution of  $M$ . However,  $R$  has positive depth and hence

$$\text{Im}(\varphi_{i+1}) = H_{\mathfrak{m}}^0(\text{Im}(\varphi_{i+1})) \subseteq H_{\mathfrak{m}}^0(G_i) = 0,$$

since  $G_i$  is a free module. Thus  $\text{Im}(\varphi_{i+1}) = 0$  and  $\text{pd}_R(M) < \infty$ . Finally, if (iii) holds, we have  $\text{Tor}_i^R(M, {}^e R) = 0$  for all  $i \geq 1$  and  $e \geq 0$  [PS73, Théorème 1.7]. In particular,  $\beta_i^F(M, {}^e R) = 0$  for all  $i \geq 1$ .  $\square$

**Corollary 4.9.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional local ring of characteristic  $p > 0$ , and let  $M$  be a finite length  $R$ -module. If  $\beta_i^F(M, R) = \beta_{i+1}^F(M, R) = 0$  for some  $i \geq 1$ , then  $\text{pd}_R(M) < \infty$ . In particular, for any parameter  $x$ , if  $\beta_2^F(R/(x), R) = 0$  then  $R$  is Cohen-Macaulay.

*Proof.* Let  $(G_j, \varphi_j)_{j \geq 0}$  be a minimal free resolution of  $M$ . Since  $\beta_i^F(M, R) = 0$ , we have that  $\text{Im}(\varphi_{i+1})$  has finite length, and it is nilpotent. Take  $q = p^e \gg 0$  so that  $\text{Im}(\varphi_{i+1}^{[q]}) = 0$ . For such  $q$  we have  $\text{Ker}(\varphi_{i+1}) = G_{i+1}$ . Since the resolution is minimal, we get

$$\lambda(\text{Tor}_{i+1}^R(M, {}^eR)) = q^\alpha \lambda\left(\frac{G_{i+1}}{\text{Im}(\varphi_{i+2}^{[q]})}\right) \geq q^\alpha \lambda\left(\frac{R}{\mathfrak{m}^{[q]}}\right),$$

where the last inequality comes from projecting onto one of the components of  $G_{i+1}$ . Dividing by  $q$  and taking limits, we get

$$\beta_{i+1}^F(M, R) = \lim_{e \rightarrow \infty} \frac{\lambda(\text{Tor}_{i+1}^R(M, {}^eR))}{q^{(1+\alpha)}} \geq \lim_{e \rightarrow \infty} \frac{\lambda(R/\mathfrak{m}^{[q]})}{q} = e_{HK}(\mathfrak{m}, R) > 0,$$

which is a contradiction.

The last claim follows from the fact that for any parameter  $x$ , we have

$$\beta_1^F(R/(x), R) \leq \lim_{e \rightarrow \infty} \frac{\lambda(H_1(x^q; R))}{q} = 0,$$

where  $H_1$  denotes the first Koszul homology (see [Rob87] and [HH93, Theorem 6.2]).  $\square$

**Lemma 4.10.** Let  $(R, \mathfrak{m}, K)$  be a local ring of positive characteristic  $p > 0$ , and  $\mathfrak{p} \in \text{Spec}(R)$ . If  $\text{pd}_R(\mathfrak{p}) < \infty$ , then  $R$  is a domain.

*Proof.* Since  $\mathfrak{p}$  has finite projective dimension, given a minimal free resolution

$$0 \rightarrow L_t \xrightarrow{\psi_t} \dots \rightarrow L_0 \rightarrow R/\mathfrak{p} \rightarrow 0$$

of  $R/\mathfrak{p}$  over  $R$ , we have that

$$0 \rightarrow L_t \xrightarrow{\psi_t^{[q]}} \dots \rightarrow L_0 \rightarrow R/\mathfrak{p}^{[q]} \rightarrow 0$$

is a minimal free resolution of  $R/\mathfrak{p}^{[q]}$  over  $R$  [PS73, Exemples 1.3 d)]. Then  $\text{Ass}_R(R/\mathfrak{p}^{[q]}) = \{\mathfrak{p}\}$ , and so,  $\mathfrak{p}^{[q]}$  is  $\mathfrak{p}$ -primary for all  $q = p^e$ . Let  $x \notin \mathfrak{p}$  and assume  $xy = 0$  for  $y \in R$ . This implies that for any  $q$ , we have  $xy \in \mathfrak{p}^{[q]}$ . We conclude that  $y \in \mathfrak{p}^{[q]}$  since  $x \notin \mathfrak{p}$ . Thus

$$y \in \bigcap_{q \geq 1} \mathfrak{p}^{[q]} = (0).$$

In particular, the localization map  $R \rightarrow R_{\mathfrak{p}}$  is injective. We have that  $\text{pd}_R(R/\mathfrak{p}) < \infty$  implies  $\text{pd}_{R_{\mathfrak{p}}}(k(\mathfrak{p})) < \infty$ . Then,  $R_{\mathfrak{p}}$  is a regular local ring; in particular, a domain. Therefore,  $R$  is a domain.  $\square$

**Proposition 4.11.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional local ring of characteristic  $p > 0$ , and  $I$  be an  $\mathfrak{m}$ -primary integrally closed ideal. If  $\beta_i^F(R/I, R) = 0$  for some  $i > 0$ , then  $R$  is regular.

*Proof.* Let  $\mathfrak{p}$  be a minimal prime of  $R$ . Since  $\beta_i^F(R/I, R) = 0$ , by Theorem 4.7 we have that  $\text{Tor}_i^R(R/I, R/\mathfrak{p}) = 0$ . By Proposition 4.3, it follows that  $\text{pd}_R(R/\mathfrak{p}) < \infty$ , and thus  $R$  is a domain by Lemma 4.10. Since one-dimensional local domains are Cohen-Macaulay, by Corollary 4.8 we have that  $\text{pd}_R(R/I) < \infty$ . In particular  $\text{Tor}_j^R(R/I, K) = 0$  for  $j \gg 0$ . We conclude that  $\text{pd}_R(K) < \infty$  because  $R/I$  tests finite projective dimension [Bur68, Theorem 5(ii)]. Hence,  $R$  is regular.  $\square$

## 5. SYZYGIES OF FINITE LENGTH

We now present several characteristic-free results. In particular, we do not always assume that the rings have positive characteristic. We focus on Question 1.2. Specifically, we give support to the claim that a finite length  $R$ -module  $M$  of infinite projective dimension cannot have a finite length syzygy  $\Omega_i$  for  $i > \dim(R) + 1$ . As a consequence of our methods, we describe, in some cases, the dimension of the syzygies.

It follows from Theorem 4.7 that if  $\dim(R) = 1$  and  $R$  has positive characteristic, then a positive answer to Question 1.2 is equivalent to the statement: for every  $M$  of finite length,  $\beta_i^F(M, R) = 0$  for some  $i > 1$  implies  $\text{pd}_R(M) < \infty$ .

We now provide an example that shows that the requirement of  $i > \dim(R) + 1$  in Question 1.2 is necessary to have a positive answer.

**Example 5.1.** Let  $R = \mathbb{F}_p[[x, y]]/(x^2, xy)$  and  $M = R/(x)$ . Then  $\dim(R) = 1$ . In addition,  $\text{pd}_R(M) = \infty$  because  $R$  is not Cohen-Macaulay. We have that  $\Omega_2 \cong H_{(x,y)}^0(R) = (x)$  has finite length.

**Lemma 5.2.** Let  $(R, \mathfrak{m}, K)$  be a local ring and let  $M$  be a finite length  $R$ -module that has a finite length syzygy  $\Omega_{i+1}$ , for some fixed  $i > 0$ . Then,

$$\text{Tor}_i^R(M, R/H_{\mathfrak{m}}^0(R)) = 0.$$

If  $R$  has positive characteristic  $p$ , then for all  $e \geq 0$

$$\text{Tor}_i^R(M, {}^e(R/H_{\mathfrak{m}}^0(R))) = 0.$$

*Proof.* Set  $H := H_{\mathfrak{m}}^0(R)$ . Let  $(G_{\bullet}, \varphi_{\bullet})$  be a minimal free resolution of  $M$ :

$$G_{\bullet} : \dots \rightarrow G_{i+1} \xrightarrow{\varphi_{i+1}} G_i \xrightarrow{\varphi_i} G_{i-1} \xrightarrow{\varphi_{i-1}} G_{i-2} \rightarrow \dots \rightarrow G_0 \rightarrow M \rightarrow 0.$$

Tensor  $G_{\bullet}$  with  $R/H$  and denote by  $\overline{G}_{\bullet}$  its residue class modulo  $H$ :

$$\overline{G}_{i+1} \xrightarrow{\overline{\varphi}_{i+1}} \overline{G}_i \xrightarrow{\overline{\varphi}_i} \overline{G}_{i-1}$$

Since  $\text{Im}(\varphi_{i+1}) = \Omega_{i+1}$  has finite length by assumption, we have  $\overline{\varphi}_{i+1} = 0$ . We want to show that  $\text{Ker}(\overline{\varphi}_i) = 0$  as well. For any  $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ , the complex  $(G_{\bullet})_{\mathfrak{p}}$  is split exact:

$$0 \longrightarrow (G_i)_{\mathfrak{p}} \xrightarrow{(\varphi_i)_{\mathfrak{p}}} (G_{i-1})_{\mathfrak{p}} \xrightarrow{(\varphi_{i-1})_{\mathfrak{p}}} (G_{i-2})_{\mathfrak{p}} \rightarrow \dots \rightarrow (G_0)_{\mathfrak{p}} \longrightarrow 0,$$

because  $M$  and  $\Omega_{i+1}$  have finite length. We have that  $\text{rk}((\varphi_i)_{\mathfrak{p}})$  is maximal, because  $\text{rk}(G_i) \leq \text{rk}(G_{i-1})$  as the localized complex is split exact, and localizing only decreases the rank of a map. Thus,  $r := \text{rk}(G_i) = \text{rk}((\varphi_i)_{\mathfrak{p}}) = \text{rk}(\varphi_i)$ . Furthermore,  $I_r(\varphi_i) \not\subseteq \mathfrak{p}$ , by split exactness. Since this holds for all  $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$ ; in particular, we have  $\text{depth}(I_r(\overline{\varphi}_i)) \geq 1$ . By the Buchsbaum-Eisenbud Criterion, we have that

$$0 \rightarrow \overline{G}_i \xrightarrow{\overline{\varphi}_i} \overline{G}_{i-1} \longrightarrow \overline{\Omega}_{i-1} = \Omega_{i-1}/H\Omega_{i-1} \rightarrow 0.$$

is an exact complex. Therefore  $\text{Ker}(\overline{\varphi}_i) = 0$ , and hence  $\text{Tor}_i^R(M, R/H) = 0$ . For the second part of the Lemma, when  $R$  has positive characteristic, the argument is the same: just notice that the complex  ${}^e(G_{\bullet})_{\mathfrak{p}}$  is again split exact for all primes  $\mathfrak{p} \neq \mathfrak{m}$  and apply the same argument as above to the map  $\overline{\varphi}_i^{[q]}$ .  $\square$

We now give results that support an affirmative answer to Question 1.2 for one-dimensional rings. Over Buchsbaum rings, the modules  $H_m^i(R)$  are  $K$ -vector spaces for  $i < \dim(R)$ . Because of this fact, we can prove the following proposition using Lemma 5.2.

**Proposition 5.3.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional Buchsbaum ring. Then the answer to Question 1.2 is positive.

*Proof.* Assume that there exists a finite length  $R$ -module  $M$  such that  $\Omega_{i+1}(M)$  has finite length for some  $i \geq 2$ . By Lemma 5.2, we have

$$0 = \operatorname{Tor}_i^R(M, R/H_m^0(R)) \cong \operatorname{Tor}_{i-1}^R(M, H_m^0(R)),$$

where  $i-1 \geq 1$  for dimension shifting. By Remark 2.5, we have that  $H_m^0(R) \cong \bigoplus_{j=1}^t K$ . Therefore,

$$0 = \operatorname{Tor}_{i-1}^R(M, H_m^0(R)) = \bigoplus_{j=1}^t \operatorname{Tor}_{i-1}^R(M, K),$$

which implies  $\operatorname{Tor}_{i-1}^R(M, K) = 0$ . Hence,  $\operatorname{pd}_R(M) \leq i-2$ .  $\square$

We now present two results about the dimension of syzygies of a finite-length module. These results will be used in Proposition 5.7 to give a case in which a finite-length module cannot have infinitely many syzygies of finite length.

**Proposition 5.4.** Let  $(R, \mathfrak{m}, K)$  be a local ring of dimension  $d$  and let  $M$  be a finite length  $R$ -module. Let  $i \geq 1$  and let  $\Omega_i$  be the  $i$ -th syzygy of  $M$ . Then either  $\dim(\Omega_i) = d$  or  $\Omega_i$  has finite length.

*Proof.* By way of contradiction, we suppose  $\dim(\Omega_i) = k$  with  $0 < k < d$ . Let  $G_\bullet \rightarrow M \rightarrow 0$  be a minimal free resolution of  $M$ . By our assumption on  $\dim(\Omega_i)$ , we can choose  $\mathfrak{p} \in \operatorname{Min}(\operatorname{ann}(\Omega_i)) \setminus (\{\mathfrak{m}\} \cup \operatorname{Min}(R))$  and localize  $G_\bullet$  at  $\mathfrak{p}$ . The resulting complex is split exact, because  $M_{\mathfrak{p}} = 0$ . In particular,  $(\Omega_i)_{\mathfrak{p}}$  is a free  $R_{\mathfrak{p}}$ -module. By our choice of  $\mathfrak{p}$ , we have that  $(\Omega_i)_{\mathfrak{p}}$  has finite length, and  $\dim(R_{\mathfrak{p}}) > 0$ , a contradiction.  $\square$

**Proposition 5.5.** Let  $(R, \mathfrak{m}, K)$  be a local ring of positive dimension. Suppose there exists an  $R$ -module  $M$  of infinite projective dimension and finite length which has a finite length syzygy  $\Omega_{i+1}$ , for some fixed  $i > 0$ . If  $\beta_i(M) \geq \beta_{i-1}(M)$ , then  $\Omega_{i-1}$  has finite length as well and  $R$  is one-dimensional.

*Proof.* Let  $(G_\bullet, \varphi_\bullet)$  be a minimal free resolution of  $M$ :

$$\begin{array}{ccccccc} G_\bullet : G_{i+1} & \xrightarrow{\varphi_{i+1}} & R^{\beta_i(M)} & \xrightarrow{\varphi_i} & R^{\beta_{i-1}(M)} & \xrightarrow{\varphi_{i-1}} & G_{i-2} \rightarrow \dots \rightarrow G_0 \rightarrow M \rightarrow 0. \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\ & \Omega_{i+1} & & \Omega_i & & \Omega_{i-1} & \end{array}$$

Let  $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ . We localize  $G_\bullet$  at  $\mathfrak{p}$ . Since both  $M$  and  $\Omega_{i+1}$  have finite length, we have a split exact sequence

$$\begin{array}{ccccccc} G_\bullet : 0 & \longrightarrow & R_{\mathfrak{p}}^{\beta_i(M)} & \longrightarrow & R_{\mathfrak{p}}^{\beta_{i-1}(M)} & \longrightarrow & (G_{i-2})_{\mathfrak{p}} \rightarrow \dots \rightarrow (G_0)_{\mathfrak{p}} \longrightarrow 0 \\ & & \searrow \cong & & \searrow & & \\ & & (\Omega_i)_{\mathfrak{p}} & & (\Omega_{i-1})_{\mathfrak{p}} & & \end{array}$$

In particular, this implies  $\beta_i(M) \leq \beta_{i-1}(M)$ . Since the opposite inequality holds by our assumption, we have equality. Set  $\beta = \beta_i(M) = \beta_{i-1}(M)$ . From the split exact sequence

above, we also get that  $R_{\mathfrak{p}}^{\beta} \cong (\Omega_i)_{\mathfrak{p}}$ ; therefore,  $(\Omega_{i-1})_{\mathfrak{p}} = 0$ . Since  $\mathfrak{p}$  is an arbitrary prime in  $\text{Spec}(R) \setminus \{\mathfrak{m}\}$ , we have that  $\Omega_{i-1}$  has finite length. Thus, we have a free complex  $0 \rightarrow F_1 = R^{\beta} \rightarrow F_0 = R^{\beta} \rightarrow 0$  with finite length homology. We conclude that  $R$  has dimension one by the New Intersection Theorem [Rob87].  $\square$

**Remark 5.6.** If in Proposition 5.5 one assumes that the sequence of Betti numbers  $\{\beta_i(M)\}$  is non-decreasing, then one can repeat the argument above to show that  $i$  is necessarily odd, and  $\beta_i(M) = \beta_{i-1}(M)$ ,  $\beta_{i-2}(M) = \beta_{i-3}(M)$ ,  $\dots$ ,  $\beta_1(M) = \beta_0(M)$ . In addition,  $\Omega_j(M)$  has finite length for all  $j$  even,  $0 \leq j \leq i+1$ . In particular, the typical situation to study would be  $(R, \mathfrak{m}, K)$  a one-dimensional ring and a resolution

$$0 \rightarrow \Omega_4 \rightarrow R^{\beta} \rightarrow R^{\beta} \begin{array}{c} \xrightarrow{\quad} \\ \searrow \quad \nearrow \\ \Omega_2 \end{array} R^{\alpha} \rightarrow R^{\alpha} \rightarrow M \rightarrow 0$$

with  $\Omega_4$  and  $\Omega_2$  of finite length.

As a consequence of these results, we give a partial answer to Question 1.2 in the case where  $M$  has eventually non-decreasing Betti numbers. It is a conjecture of Avramov that every finitely generated module over a local ring has eventually non-decreasing Betti numbers [Avr84]. The conjecture is known to be true in several cases [Eis80, Les90, Cho90, Sun94, AGP97, Sun98], in particular, for Golod rings [Les90, Corollaire 6.5].

**Proposition 5.7.** Let  $(R, \mathfrak{m}, K)$  be a local ring and let  $M$  be a finite length  $R$ -module of infinite projective dimension with eventually non-decreasing Betti numbers. Then, for all  $i \gg 0$ , there exists  $\mathfrak{p} \in \text{Min}(R)$  such that  $\dim(\Omega_i) = \dim(R/\mathfrak{p})$ . In particular,  $M$  cannot have arbitrarily high syzygies of finite length.

*Proof.* If  $\text{Supp}(\Omega_i) \cap \text{Min}(R) \neq \emptyset$  for all  $i \gg 0$  then we are done. By way of contradiction, assume that there exist infinitely many syzygies  $\Omega_i$  of  $M$  such that  $\text{Supp}(\Omega_i) \cap \text{Min}(R) = \emptyset$ . Notice that, by Proposition 5.4, such syzygies must have finite length. By replacing  $M$  with a high enough syzygy, we can then assume that  $M$  is a module of finite length with non-decreasing Betti numbers, and with infinitely many syzygies of finite length. We have that  $R$  is one-dimensional by Proposition 5.5. Furthermore, by Remark 5.6, we have  $\beta_{2i} = \beta_{2i+1}$  for all  $i \geq 0$ . For  $i \geq 0$  consider the short exact sequence

$$0 \rightarrow \Omega_{2i+2} \rightarrow R^{\beta} \xrightarrow{\varphi} R^{\beta} \rightarrow \Omega_{2i} \rightarrow 0,$$

where  $\beta := \beta_{2i} = \beta_{2i+1}$ . Let  $S := R[\varphi]$ . Then  $R^{\beta}$  becomes an  $S$ -module. The exact sequence above shows that  $\Omega_{2i} \cong R^{\beta} \otimes_S S/(\varphi)$  and  $\Omega_{2i+2} \cong (0 :_{R^{\beta}} \varphi)$ . Then, by [SH06, Proposition 11.1.9 (2)],

$$\lambda(\Omega_{2i}) - \lambda(\Omega_{2i+2}) = e(\varphi; R^{\beta}),$$

where  $e(\varphi; -)$  denotes the Hilbert-Samuel multiplicity with respect to the ideal  $(\varphi)$  in  $S$ . Since such multiplicity is always positive, we have that  $\lambda(\Omega_{2i+2}) < \lambda(\Omega_{2i})$ , for all  $i \geq 0$ . Since there cannot be an infinite strictly decreasing sequence of such lengths, we obtain a contradiction.  $\square$

**Remark 5.8.** Proposition 5.7 also follows from [BL13, Theorem 8], and it gives another proof of the fact that, when  $M$  is a module of finite length with eventually non-decreasing Betti numbers and  $R$  is equidimensional, then the sequence of integers  $\{\dim(\Omega_i)\}_{i=0}^{\infty}$  is constant for  $i \gg 0$  (see [BL13, Corollary 2]).



**Proposition 5.9.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional local ring. Suppose that there exists a finite length module  $M$  of infinite projective dimension that has a finite length syzygy  $\Omega_{i+1}$ , for some fixed  $i \geq 2$ . Then

$$\lambda(\Omega_{i+1}) = \sum_{j=0}^i (-1)^{i-j+1} \lambda(\operatorname{Tor}_j^R(M, R/(x))),$$

where  $x$  is a suitable parameter.

*Proof.* Consider a minimal free resolution of  $M$ :

$$G_{\bullet} : \quad \dots \rightarrow G_{i+1} \xrightarrow{\quad} G_i \xrightarrow{\quad} G_{i-1} \rightarrow \dots \rightarrow G_1 \xrightarrow{\quad} G_0 \rightarrow M \rightarrow 0.$$

$\searrow \quad \nearrow \quad \searrow \quad \nearrow \quad \searrow \quad \nearrow$   
 $\Omega_{i+1} \quad \Omega_i \quad \Omega_1$

For all  $j = 1, \dots, i+1$ , we break it into short exact sequences:

$$0 \longrightarrow \Omega_j \longrightarrow G_{j-1} \longrightarrow \Omega_{j-1} \longrightarrow 0,$$

where  $\Omega_0 := M$ . These give two exact sequences:

$$0 \longrightarrow \Omega_{i+1} \longrightarrow H_{\mathfrak{m}}^0(G_i) \longrightarrow H_{\mathfrak{m}}^0(\Omega_i) \longrightarrow 0$$

and

$$0 \longrightarrow H_{\mathfrak{m}}^0(\Omega_j) \longrightarrow H_{\mathfrak{m}}^0(G_{j-1}) \longrightarrow H_{\mathfrak{m}}^0(\Omega_{j-1}).$$

The first short exact sequence comes from the fact that  $\Omega_{i+1}$  has finite length, and so,  $H_{\mathfrak{m}}^1(\Omega_{i+1}) = 0$ . Furthermore, the cokernel of the rightmost map in the second exact sequence, which can be proved to be the kernel of the leftmost map in

$$\Omega_j \otimes_R H_{\mathfrak{m}}^1(R) \longrightarrow G_{j-1} \otimes_R H_{\mathfrak{m}}^1(R) \longrightarrow \Omega_{j-1} \otimes_R H_{\mathfrak{m}}^1(R) \longrightarrow 0$$

is then  $\operatorname{Tor}_1^R(\Omega_{j-1}, H_{\mathfrak{m}}^1(R))$ . For simplicity, we denote  $\omega_j := \lambda(H_{\mathfrak{m}}^0(\Omega_j))$ ,  $g_j := \lambda(H_{\mathfrak{m}}^0(G_j))$  and  $\alpha_j := \lambda(\operatorname{Tor}_1^R(\Omega_j, H^1(R)))$ . Then, we have relations

$$\begin{aligned} \omega_{i+1} &= g_i - \omega_i \\ \omega_i &= g_{i-1} - \omega_{i-1} + \alpha_{i-1} \\ &\vdots \\ \omega_2 &= g_1 - \omega_1 + \alpha_1 \\ \omega_1 &= g_0 - \lambda(M) + \lambda(\operatorname{Tor}_1(M, H_{\mathfrak{m}}^1(R))). \end{aligned}$$

After localizing the resolution  $G_{\bullet}$  at any minimal prime  $\mathfrak{p}$ , since  $(\Omega_{i+1})_{\mathfrak{p}} = 0$ , we obtain that  $\sum_{j=0}^i (-1)^j \beta_j(M) = 0$ . Then,  $\sum_{j=0}^i (-1)^j g_j = 0$  because  $g_j = \beta_j(M) \cdot \lambda(H_{\mathfrak{m}}^0(R))$ . Therefore,

$$\omega_{i+1} = \lambda(\Omega_{i+1}) = \sum_{j=1}^{i-1} (-1)^{i-j} \alpha_j + (-1)^i \lambda(\operatorname{Tor}_1(M, H^1(R))) + (-1)^{i-1} \lambda(M).$$

Choose a parameter  $x$  such that  $H_{\mathfrak{m}}^0(R) = 0 :_R x$ , as in Remark 2.4. By similar considerations we can also assume that  $xM = 0$ . From this choice, we have that  $xH_{\mathfrak{m}}^0(\Omega_j) = 0$  for all  $j = 0, \dots, i+1$ , because  $\Omega_j \subseteq G_{j-1}$  is a free  $R$ -module. Since the Tor modules can be computed using flat resolutions, we have an exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(R) \rightarrow R \rightarrow R_x \rightarrow H_{\mathfrak{m}}^1(R) \rightarrow 0.$$

We complete on the left to get a flat resolution of  $H_{\mathfrak{m}}^1(R)$ :

$$\begin{array}{ccccccc} \dots & \longrightarrow & R^{\mu(H_{\mathfrak{m}}^0(R))} & \longrightarrow & R & \xrightarrow{\quad} & R_x \longrightarrow H_{\mathfrak{m}}^1(R) \longrightarrow 0 \\ & & & & & \searrow & \nearrow \\ & & & & & R/H_{\mathfrak{m}}^0(R) & \end{array}$$

By our choice of  $x$ , we have that a free resolution of  $R/x$  starts as

$$\dots \longrightarrow R^{\mu(H_{\mathfrak{m}}^0(R))} \longrightarrow R \longrightarrow R \longrightarrow R/(x) \longrightarrow 0.$$

For all  $j = 1, \dots, i-1$ , we obtain

$$\mathrm{Tor}_1^R(\Omega_j, H_{\mathfrak{m}}^1(R)) \cong \mathrm{Tor}_1^R(\Omega_j, R/(x)) \cong \mathrm{Tor}_{j+1}^R(M, R/(x)),$$

where the last isomorphism comes from dimension shifting. In addition,

$$\mathrm{Tor}_1^R(M, H_{\mathfrak{m}}^1(R)) \cong \mathrm{Tor}_1^R(M, R/(x)).$$

Finally, since  $xH_{\mathfrak{m}}^0(\Omega_0) = xM = 0$ , we get

$$M \cong M/xM \cong \mathrm{Tor}_0^R(M, R/(x)),$$

and the proposition then follows.  $\square$

**Corollary 5.10.** Let  $(R, \mathfrak{m}, K)$  be a one-dimensional ring, and let  $M$  be a finite length module of infinite projective dimension. Then  $\lambda(\Omega_1) = \lambda(\Omega_3) = \infty$ .

*Proof.* Note that  $\lambda(\Omega_1) = \infty$ ; otherwise, we would have a short exact sequence

$$0 \longrightarrow \Omega_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

in which both  $\Omega_1$  and  $M$  have finite length. This cannot happen because  $G_0 \neq 0$  is free and  $\dim(R) = 1$ . Now let us assume by way of contradiction that  $\lambda(\Omega_3) < \infty$ . Let  $(G_{\bullet}, \varphi_{\bullet})$  be a minimal free resolution of  $M$ :

$$0 \longrightarrow \Omega_3 \longrightarrow G_2 \xrightarrow{\varphi_2} G_1 \xrightarrow{\varphi_1} G_0 \longrightarrow M \longrightarrow 0$$

Let  $x \in R$  be a parameter such that  $xM = xH_{\mathfrak{m}}^0(R) = 0$ . Consider the short exact sequence

$$0 \longrightarrow (x) \longrightarrow R \longrightarrow R/(x) \longrightarrow 0.$$

By our choice of  $x$  we have  $0 :_R x = H_{\mathfrak{m}}^0(R)$ , hence  $(x) \cong R/H_{\mathfrak{m}}^0(R)$ . After tensoring the sequence with  $M$ , we obtain that

$$0 \longrightarrow \mathrm{Tor}_1^R(M, R/(x)) \longrightarrow M/H_{\mathfrak{m}}^0(R)M \longrightarrow M \longrightarrow M/xM \longrightarrow 0.$$

Since  $xM = 0$ , we obtain

$$\lambda(\mathrm{Tor}_1^R(M, R/(x))) = \lambda(M/H_{\mathfrak{m}}^0(R)M).$$

Then, by Proposition 5.9 we have

$$\begin{aligned} \lambda(\Omega_3) &= -\lambda(\mathrm{Tor}_2^R(M, R/(x))) + \lambda(\mathrm{Tor}_1^R(M, R/(x))) - \lambda(M) \\ &\leq \lambda(\mathrm{Tor}_1^R(M, R/(x))) - \lambda(M) \\ &= \lambda(M/H_{\mathfrak{m}}^0(R)M) - \lambda(M) \leq 0, \end{aligned}$$

which gives a contradiction since  $\Omega_3 \neq 0$ , because  $M$  has infinite projective dimension.  $\square$

The following example is due to the second author, and it is taken from [BL13]. It shows the assumption that  $M$  has finite length is needed in Corollary 5.10.

**Example 5.11.** Let  $S = \mathbb{Q}[x, y, z, u, v]$  and let  $I \subseteq S$  be the ideal

$$I = (x^2, xz, z^2, xu, zv, u^2, v^2, zu + xv + uv, yu, yv, yx - zu, yz - xv).$$

Let  $R = S/I$ , which is a one-dimensional ring of depth 0. In this case  $y$  is a parameter,  $0 :_R y = (u, v, z^2)$  and  $(y) = 0 :_R (0 :_R y)$ . Let  $M$  be the cokernel of the rightmost map in the following exact complex

$$\dots \longrightarrow R^3 \xrightarrow{\begin{bmatrix} u & v & z^2 \end{bmatrix}} R \xrightarrow{y} R \xrightarrow{\begin{bmatrix} u \\ v \\ z^2 \end{bmatrix}} R^3.$$

Then  $M$  is a one-dimensional module with first and third syzygies  $\Omega_1 \cong R/(y)$  and  $\Omega_3 \cong 0 :_R y$ . These are both modules of finite length since  $y$  is a parameter.

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